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Fresnel Integral Equations: Numerical Properties

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Abstract — **A spatial-domain solution to the problem of electromagnetic scattering from a dielectric half-space is outlined. The resulting half-space operators are referred to as Fresnel surface integral operators. When used as preconditioners for nonplanar geometries, the Fresnel operators yield surface Fresnel integral equations (FIEs) which are stable with respect to dielectric constant, discretization, and frequency. Numerical properties of the formulations are discussed.**

1 Introduction

A common integral equation formulation of electromagnetic scattering from dielectric interfaces is the PMCHW formulation (after Poggio, Miller, Chang, Harrington and Wu). Unfortunately, standard numerical discretizations of the PMCHW formulation do not yield well-conditioned matrix equations. Motivated by similar efforts for scattering from perfect electric conductors [1], we seek a preconditioner for the PMCHW formulation of the dielectric scattering problem which renders the formulation stable with respect to frequency, discretization, and dielectric contrast ratio. For reasons that are explained below, the resulting preconditioning operator for the coupled PMCHW equations is referred to as the Fresnel matrix. The Fresnel matrix is determined by inverting the PMCHW formulation for scattering from a dielectric half-space in the spatial-domain. Unlike previous solutions to the dielectric half-space scattering problem, the resulting inverse operator (i.e., the Fresnel matrix) is expressed in terms of the traditional surface integral operators of vector and scalar scattering theory. Because the Fresnel matrix is defined entirely in terms of familiar, spatial-domain operators, it is trivial to apply the Fresnel matrix as an analytic preconditioner of the PMCHW formulation of scattering from nonplanar dielectric surfaces. The resulting equations, referred to as Fresnel integral equations (FIEs), are defined below. In contrast to the standard PMCHW formulation, the FIE formulation is stable with respect to the dielectric constant, the discretization mesh, and the frequency for scattering from smooth, nonplanar dielectric interfaces. Preliminary numerical comparisons are provided.

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2 Dielectric Formulations

Consider the dielectric half-space scattering problem. Using the notation defined in [1], the BIEs in the upper (subscript 1) and lower (subscript 2) media are

$$\begin{aligned} \frac{1}{2}\boldsymbol{J}_1 &= \boldsymbol{J}_1^i - K_1\boldsymbol{J}_1 + \eta_1^{-1}T_1\boldsymbol{M}_1 \\ \frac{1}{2}\boldsymbol{M}_1 &= \boldsymbol{M}_1^i + \eta_1 T_1\boldsymbol{J}_1 - K_1\boldsymbol{M}_1 \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{1}{2}\boldsymbol{J}_2 &= \boldsymbol{J}_2^i + K_2\boldsymbol{J}_2 - \eta_2^{-1}T_2\boldsymbol{M}_2 \\ \frac{1}{2}\boldsymbol{M}_2 &= \boldsymbol{M}_2^i - \eta_2 T_2\boldsymbol{J}_2 + K_2\boldsymbol{M}_2 \end{aligned} \tag{2}$$

where $\boldsymbol{J}_1 = \hat{\boldsymbol{n}}\times\boldsymbol{H}_1$, $\boldsymbol{M}_1 = -\hat{\boldsymbol{n}}\times\boldsymbol{E}_1$, $\boldsymbol{J}_2 = \hat{\boldsymbol{n}}\times\boldsymbol{H}_2$, $\boldsymbol{M}_2 = -\hat{\boldsymbol{n}}\times\boldsymbol{E}_2$ and

$$\begin{aligned} T_1\boldsymbol{J} &= jk_1\hat{\boldsymbol{n}}\times\int_S \boldsymbol{J}(\boldsymbol{r}')G_1(\boldsymbol{r},\boldsymbol{r}')\,ds' \\ &\quad - \frac{1}{jk_1}\hat{\boldsymbol{n}}\times \fint_S \boldsymbol{\nabla}'\cdot\boldsymbol{J}_1(\boldsymbol{r}')\boldsymbol{\nabla}G_1(\boldsymbol{r},\boldsymbol{r}')\,ds' \\ K_1\boldsymbol{J} &= \hat{\boldsymbol{n}}\times\int_S \boldsymbol{J}(\boldsymbol{r}')\times\boldsymbol{\nabla}G_1(\boldsymbol{r},\boldsymbol{r}')\,ds' \end{aligned} \tag{3}$$

G_1 and G_2 are the Green functions in the homogeneous media defined by k_1 and k_2. The local normal vector $\hat{\boldsymbol{n}}$ always points from the homogeneous region defined by k_2 into the homogeneous region defined by k_1. The definitions of T_2 and K_2 are obtained from (3) after an appropriate change of subscripts. Continuity of tangential $\boldsymbol{E}$ and $\boldsymbol{H}$ fields accross the shared boundary provides the conditions

$$\boldsymbol{J}_2 = \boldsymbol{J}_1 \qquad \boldsymbol{M}_2 = \boldsymbol{M}_1 \tag{4}$$

With (4) the integral equations (2) are

$$\begin{aligned} \frac{1}{2}\boldsymbol{J}_1 &= \boldsymbol{J}_2^i + K_2\boldsymbol{J}_1 - \eta_2^{-1}T_2\boldsymbol{M}_1 \\ \frac{1}{2}\boldsymbol{M}_1 &= \boldsymbol{M}_2^i - \eta_2 T_2\boldsymbol{J}_1 + K_2\boldsymbol{M}_1 \end{aligned} \tag{5}$$

Subtracting the equations for $\boldsymbol{J}_1$ from (1) and (5) yields the single condition

$$0 = \boldsymbol{J}^i - (K_1+K_2)\boldsymbol{J}_1 + \eta_1^{-1}(T_1 + r^{-1}T_2)\boldsymbol{M}_1 \tag{6}$$

where $\boldsymbol{J}^i = \boldsymbol{J}_1^i - \boldsymbol{J}_2^i$ and $r = \eta_2/\eta_1$. The equations for $\boldsymbol{M}_1$ provide the additional condition

$$0 = \boldsymbol{M}^i + \eta_1(T_1 + rT_2)\boldsymbol{J}_1 - (K_1 + K_2)\boldsymbol{M}_1 \tag{7}$$

where $\boldsymbol{M}^i = \boldsymbol{M}_1^i - \boldsymbol{M}_2^i$.

Equations (7) and (6) combine to form the simultaneous system

$$\bar{Q}\begin{bmatrix}\boldsymbol{J}_1\\ \boldsymbol{M}_1\end{bmatrix} = \begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix} \tag{8}$$

where

$$\bar{Q} = \begin{bmatrix} K_1 + K_2 & -\eta_1^{-1}(T_1 + r^{-1}T_2) \\ -\eta_1(T_1 + rT_2) & K_1 + K_2 \end{bmatrix} \tag{9}$$

An overbar is used to indicate that $\bar{Q}$ is a two-by-two matrix of vector operators.

2.1 Half-space problem

The simultaneous system of integral equations (8) provides a solvable boundary integral equation formulation of scattering from an arbitrary dielectric interface in three dimensions. For the planar dielectric half-space problem $K \equiv 0$. In this case (8) reduces to

$$\bar{Q}_0\begin{bmatrix}\boldsymbol{J}_1\\ \boldsymbol{M}_1\end{bmatrix} = \begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix} \tag{10}$$

where

$$\bar{Q}_0 = \begin{bmatrix} 0 & -\eta_1^{-1}(T_1 + r^{-1}T_2) \\ -\eta_1(T_1 + rT_2) & 0 \end{bmatrix} \tag{11}$$

We also define a related half-space operator, $\bar{Q}_t$, which will be useful in determining a solution to (10),

$$\bar{Q}_t = \begin{bmatrix} 0 & -\eta_1^{-1}(T_1 + rT_2) \\ -\eta_1(T_1 + r^{-1}T_2) & 0 \end{bmatrix} \tag{12}$$

3 Direct Solution of Half-space Problem

In this section we determine a direct expression for the Fresnel matrix, $\bar{\Gamma}$, which determines the total half-space surface currents $\boldsymbol{J}_1$ and $\boldsymbol{M}_1$ from the incident quantities $\boldsymbol{J}^i$ and $\boldsymbol{M}^i$,

$$\begin{bmatrix}\boldsymbol{J}_1\\ \boldsymbol{M}_1\end{bmatrix} = \bar{\Gamma}\begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix} \tag{13}$$

Comparing this with (10) indicates that $\bar{\Gamma} = \bar{Q}_0^{-1}$.

The determination of a direct form for the inverse of $\bar{Q}_0$ is complicated relative to the scalar half-space problem (previously considered elsewhere) by its vector nature. The component operators of $\bar{Q}_0$ (T_1 and T_2) couple the rotational and irrotational subspaces of $\boldsymbol{J}_1$ and $\boldsymbol{M}_1$ at the half-space interface [1]. As shown below, this complication is removed after multiplication of (10) by $\bar{Q}_t$,

$$\bar{Q}_t\bar{Q}_0\begin{bmatrix}\boldsymbol{J}_1\\ \boldsymbol{M}_1\end{bmatrix} = \bar{Q}_t\begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix} \tag{14}$$

The product on the left side of this equation has the form

$$\bar{Q}_t\bar{Q}_0 = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \tag{15}$$

where

$$U_1 = -\frac{1}{4}(1 + r^2) + r(T_1T_2 + T_2T_1) \tag{16}$$

$$U_2 = -\frac{1}{4}(1 + r^{-2}) + r^{-1}(T_1T_2 + T_2T_1) \tag{17}$$

and the half-space identity

$$T_i^2 = -\frac{1}{4} \tag{18}$$

was used.

3.1 Vector and scalar integral operators

Inversion of (14) relies on the following identity which relates the electromagnetic operators T_1 and T_2 over a planar interface to the Dirichlet-to-Neumann ($\mathcal{N}$) and Neumann-to-Dirichlet ($\mathcal{D}$) operators of scalar scattering theory:

$$T_1T_2 + T_2T_1 = -4\frac{r^2+1}{r}\mathcal{N}_3^2\mathcal{D}_1\mathcal{D}_2 \tag{19}$$

The essential properties of $\mathcal{N}$ and $\mathcal{D}$ have been discussed elsewhere and are breifly reviewed below. The half-space identity (19) provides a connection between the previously discussed scalar half-space problem and the vector electromagnetic problem examined here.

Operators $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined in the respective homogeneous media defined by ε_1 and ε_2. The operator $\mathcal{N}_3$ is defined by the derived dielectric constant

$$\varepsilon_3 = \frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \tag{20}$$

On a planar interface, $\mathcal{D}_1$ and $\mathcal{N}_1$ are defined as

$$\mathcal{D}_1\boldsymbol{J} = \int_S G_1\boldsymbol{J}\,ds' \tag{21}$$

$$\mathcal{N}_1\boldsymbol{J} = \frac{\partial}{\partial n}\int_S \frac{\partial}{\partial n'}G_1\boldsymbol{J}\,ds' \tag{22}$$

Similar definitions are obtained for $\mathcal{D}_2$ and $\mathcal{N}_3$ by replacing the upper medium Green function G_1 by the Green function for the homogeneous media defined by ε_2 and ε_3, respectively. In all cases $\mathcal{N}_i$ and $\mathcal{D}_i$ satisfy the relation

$$\mathcal{D}_i\mathcal{N}_i = -1/4 \tag{23}$$

The subscript i is used to indicate that the operators $\mathcal{D}_i$ and $\mathcal{N}_i$ are defined in the medium characterized by the dielectric constant ε_i.

Substituting (19) in (16) yields

$$U_1 = -\frac{1}{4}(1+r^2) - 4(r^2+1)\mathcal{N}_3^2\mathcal{D}_1\mathcal{D}_2 \tag{24}$$

This can be rewritten using identity (23) as

$$U_1 = -4(1+r^2)\mathcal{N}_2\mathcal{N}_1\mathcal{D}_1\mathcal{D}_2 - 4(r^2+1)\mathcal{N}_3^2\mathcal{D}_1\mathcal{D}_2 \tag{25}$$

or

$$U_1 = -4(1+r^2)\left(\mathcal{N}_2\mathcal{N}_1 + \mathcal{N}_3^2\right)\mathcal{D}_1\mathcal{D}_2 \tag{26}$$

A similar procedure allows U_2 to be written

$$U_2 = -4(1+r^{-2})\left(\mathcal{N}_2\mathcal{N}_1 + \mathcal{N}_3^2\right)\mathcal{D}_1\mathcal{D}_2 \tag{27}$$

3.2 Inversion of scalar operators

Using (26) and (27) in (14) leads to

$$\begin{aligned} -4\mathcal{D}_1\mathcal{D}_2(\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_3^2)\begin{bmatrix}1+r^2 & 0\\ 0 & 1+r^{-2}\end{bmatrix}\begin{bmatrix}\boldsymbol{J}_1\\ \boldsymbol{M}_1\end{bmatrix} \\ = \bar{Q}_t\begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix}\end{aligned} \tag{28}$$

Using (23) this can be rewritten

$$(\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_3^2)\begin{bmatrix}\boldsymbol{J}_1\\ \boldsymbol{M}_1\end{bmatrix} = \frac{-4\mathcal{N}_1\mathcal{N}_2}{1+r^2}\begin{bmatrix}1 & 0\\ 0 & r^2\end{bmatrix}\bar{Q}_t\begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix} \tag{29}$$

Multiplying (29) by $(\mathcal{N}_1\mathcal{N}_2 - \mathcal{N}_3^2)$ provides

$$\begin{aligned}&(\mathcal{N}_1^2\mathcal{N}_2^2 - \mathcal{N}_3^4)\begin{bmatrix}\boldsymbol{J}_1\\ \boldsymbol{M}_1\end{bmatrix} \\ &= (\mathcal{N}_1\mathcal{N}_2 - \mathcal{N}_3^2)\frac{-4\mathcal{N}_1\mathcal{N}_2}{1+r^2}\begin{bmatrix}1 & 0\\ 0 & r^2\end{bmatrix}\bar{Q}_t\begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix}\end{aligned} \tag{30}$$

A direct expression for the surface currents is obtained by introducing a final operator identity for the half-space problem,

$$\mathcal{N}_1^2\mathcal{N}_2^2 - \mathcal{N}_3^4 = -\frac{k_2^2}{4}\frac{(r^2-1)^2}{r^2+1}\mathcal{N}_e^2 \tag{31}$$

The hypersingular operator $\mathcal{N}_e$ is defined by the effective dielectric constant

$$\varepsilon_e = \frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2} \tag{32}$$

It follows from (31) and (23) that

$$(\mathcal{N}_1^2\mathcal{N}_2^2 - \mathcal{N}_3^4)^{-1} = -16\frac{4(r^2+1)}{k_2^2(r^2-1)^2}\mathcal{D}_e^2 \tag{33}$$

3.3 Fresnel matrix

Using (33) in (30) provides a direct expression for the half-space surface currents,

$$\begin{bmatrix}\boldsymbol{J}_1\\ \boldsymbol{M}_1\end{bmatrix} = \bar{\Gamma}\begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix} \tag{34}$$

where the Fresnel matrix $\bar{\Gamma}$ is expressed in terms of standard surface integral operators as

$$\begin{aligned}\bar{\Gamma} = \frac{256}{k_2^2(r^2-1)^2}\mathcal{D}_e^2(\mathcal{N}_1\mathcal{N}_2 - \mathcal{N}_3^2)\mathcal{N}_1\mathcal{N}_2 \cdot \\ \cdot\begin{bmatrix}1 & 0\\ 0 & r^2\end{bmatrix}\bar{Q}_t\end{aligned} \tag{35}$$

or

$$\begin{aligned}\bar{\Gamma} = \frac{256}{k_2^2(r^2-1)^2}\mathcal{D}_e^2(\mathcal{N}_1\mathcal{N}_2 - \mathcal{N}_3^2)\mathcal{N}_1\mathcal{N}_2 \cdot \\ \cdot\begin{bmatrix}0 & -\eta_1^{-1}(T_1 + rT_2)\\ -\eta_1(r^2T_1 + rT_2) & 0\end{bmatrix}\end{aligned} \tag{36}$$

The Fresnel matrix of (36) is the primary analytical result of this paper. Given the incident fields generated by an arbitrary source over the surface of a dielectric half-space, $\bar{\Gamma}$ determines the total electric and magnetic equivalent currents. The solution is obtained without decomposing the incident fields into orthogonally polarized components.

Observe that, in addition to operators defined in the original media indicated by ε_1 and ε_2, the Fresnel matrix involves operators defined over two derived media, indicated by ε_3 and ε_e. The operator $\mathcal{D}_e$ is essential. It has an infinite eigenvalue at the critical angle and is thereby able to directly incorporate surface wave phenomena. In contrast, the operator $\mathcal{N}_3^2$ is nonessential. It is possible to express this operator as a weighted difference between operators $\mathcal{N}_e^2$ and $\mathcal{N}_1^2$, for example. A similar decomposition of $\mathcal{D}_e^2$ does not appear to be a possibility.

4 Fresnel Integral Equations

The Fresnel matrix has been determined by solving the planar half-space problem for an arbitrary excitation. Unlike other solutions of this problem, $\bar{\Gamma}$ is expressed in terms of standard surface integral operators. For this reason, all operators used to define $\bar{\Gamma}$ in (36) are well defined for an arbitrarily shaped interface. Consequently, $\bar{\Gamma}$ provides a useful preconditioning operator for arbitrarily shaped interfaces. Starting with the general formulation (8) we have

$$\bar{\Gamma}\bar{Q}\begin{bmatrix}\boldsymbol{J}\\ \boldsymbol{M}\end{bmatrix} = \bar{\Gamma}\begin{bmatrix}\boldsymbol{J}^i\\ \boldsymbol{M}^i\end{bmatrix} \tag{37}$$

4.1 Numerical properties: Half-space

It is evident from the preceding discussion that the condition number of the operator on the left side of (37), $\bar{\Gamma}\bar{Q}$, is unity at all frequencies for a planar interface characterized by arbitrary dielectric constants ε_1 and ε_2. In the following we consider the corresponding condition numbers of the original half-space formulation defined in (10). This is accomplished by considering a Fourier decomposition of the currents on the planar boundary, which is assumed to be the Cartesian x-y plane. The parameters k_x and k_y are used to indicate the respective wavenumbers in the x and y directions. A complete description of the continuous equations (10) and (37) requires an infinite range of these parameters. However, the numerical discretization procedure effectively truncates the range of k_x and k_y which can impact the conditioning of the discrete operator. Thus, in the following examples we model the resolution of the discretization mesh by appropriately truncating the range of k_x and k_y used to compute the condition numbers of Q_0.

Figure 1 illustrates the variation in the condition number of Q_0 as a function of the dielectric contrast ratio, $\varepsilon_2/\varepsilon_1$, where ε_1 is real and $\varepsilon_2 = \varepsilon_{2r} - j\varepsilon_{2i}$ with $\varepsilon_{2r} = \varepsilon_{2i}$. The discretization of the half-space problem is assumed to be coarse such that the maximum resolvable transverse wavenumbers k_x and k_y are $10\lambda_1^{-1}$ where λ_1 indicates the wavelength in the upper medium. The figure indicates that (i) the Fresnel integral equation (37) provides a significant improvement in the condition number for all contrast ratios, and (ii) the relative improvement provided by (37) increases as the dielectric contrast decreases.

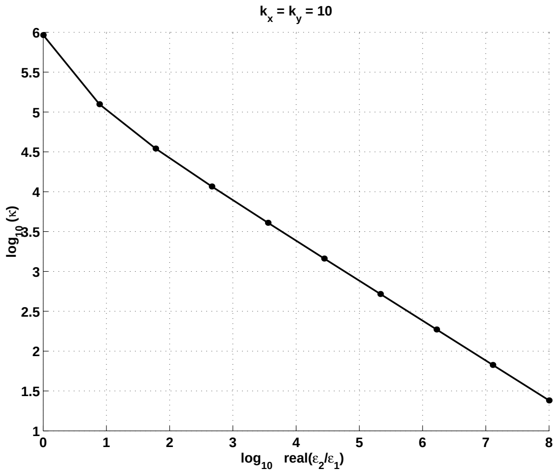

Figure 1: Variation of condition number (κ) with dielectric constant. ($k_x = k_y = 10\lambda_1^{-1}$).

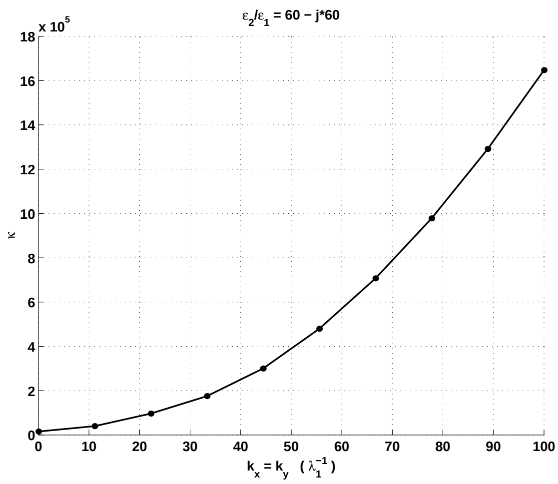

Figure 2: Variation of condition number (κ) versus discretization resolution for a fixed dielectric ratio.

Figure 2 depicts the dependence of the condition number of Q_0 in (10) on the resolution of the discretization mesh for a dielectric contrast ratio $\varepsilon_2/\varepsilon_1 = 60 - j60$. The condition number increases approximately quadratically as the mesh resolution in the x- and y-directions increases. This behavior follows from the simultaneous smoothing and differentiating properties of the operators T_1 and T_2 [1]. Finally, we observe that the condition number of (37) is unity for all mesh resolutions, providing a significant improvement over the standard formulation (10) illustrated in Figure 2.

5 Summary

Fresnel integral equations (FIEs) have been determined thru the development of a new representation for the solution to the half-space dielectric scattering problem. The new integral equations have a condition number which is independent of frequency, dielectric constants, and mesh resolution. The extension of the foregoing results to formulations of the scattering problem other than (8) will be discussed during the presentation of this paper.

References

[1] R. J. Adams, "Physical and analytical properties of a stabilized electric field integral equation," *IEEE Transactions on Antennas and Propagation*, accepted for publication.